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# SHORTER COMMUNICATIONS

## THE FREEZING OF SPHERES

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### NOMENCLATURE

| | |
|---|---|
| $B^2$, | $\equiv -(1/2\delta)y_1^3/y_2$, dimensionless; |
| $c$, | specific heat; |
| $k$, | thermal conductivity; |
| $K$, | constant value of $(d^2y/dt^2) \div (dy/dt)^3$, $K \neq 0$; |
| $L$, | latent heat of fusion; |
| $p$, | $\equiv y_0y_2/y_1^2$, dimensionless; |
| $q$, | $\equiv y_0y_1$ [area/time]; |
| $r$, | radius; |
| $r^*$, | $\equiv y^* \equiv y_0 + y_1^2/y_2$ [length]; |
| $t$, | time; |
| $T$, | time required for complete solidification to occur; |
| $U(r, t)$, | temperature at radius $r$ at time $t$[deg]; |
| $U^*$, | solidification temperature [deg]; |
| $y(t)$, | radial location of moving phase boundary [length]; |
| $y_n$, | $\equiv [d^ny(t)/dt^n]_{t=0}$ [length/(time)$^n$]; |
| $y^*$, | $\equiv y_0 + y_1^2/y_2$ [length]. |

Greek symbols

| | |
|---|---|
| $\delta$, | $\equiv k/\rho c$ = thermal diffusivity; |
| $\rho$, | density. |

### INTRODUCTION

PROBLEMS OF the solidification of spheres are of interest in foundary practice and in the freezing of raindrops, but closed form solutions to such problems are not to be found in standard texts [1]. We state the solidification problem in precise mathematical form, present an exact closed form solution for the temperature distribution, and obtain a good correlation of experimental data on freezing front motion.

### THE PROBLEM

Radial heat conduction in the frozen outer shell of a solidifying sphere is governed by the following partial differential equation:

$$(\partial/\partial r)\,\{4\pi r^2\,[-k\,(\partial U/\partial r)]\} + 4\pi r^2\,[\rho c\,(\partial U/\partial t)] = 0. \qquad (1)$$

Here $U(r, t)$ is the temperature in the frozen shell at radius $r$ at time $t$, $r \leqslant y_0$; $-k(\partial U/\partial r)$ is the heat conduction rate per unit area at radius $r$ at time $t$; $4\pi r^2$ is the surface area of a sphere of radius $r$. Solidification is assumed to occur at the constant temperature $U^*$. The radial location $r = y(t)$ of the moving solidification front is therefore defined by the implicit relation

$$U(r, t) \equiv U^* = \text{constant}, \quad \text{at} \quad r = y(t). \qquad (2)$$

Movement of a solidification front at a rate $dy/dt$ is accompanied by the release of heat energy at a rate $|(4\pi y^2)\rho L(dy/dt)|$, where $4\pi y^2$ is the surface area of the solidification front and $L$ is the latent heat of fusion. The energy released during solidification must be removed from the solidification front by heat conduction. The heat-conduction rate at radius $r$ is equal to the product of the negative temperature gradient $-\partial U/\partial r$, of the thermal conductivity $k$, and of the area $4\pi r^2$, so

$$|4\pi r^2[-k(\partial U/\partial r)]| \geqslant |(4\pi y^2)\rho L(dy/dt)|, \quad \text{at} \quad r = y(t). \qquad (3a)$$

An inequality is used because the heat-conduction rate in the solid must be large enough to allow the removal of the latent heat plus the removal of any heat conducted to the solidification front from the liquid at the center of the sphere.

Condition (3a) is now simplified by assuming that the entire sphere is initially composed of liquid which is at the solidification temperature $U^*$, this corresponding to the absence of superheat. With this assumption, the temperature is always equal to $U^*$ in the liquid region $r < y(t)$. There can then be no heat conduction from the isothermal liquid to the solidification front so condition (3a) becomes an equality. The equality can be divided by $4\pi y^2$ to obtain

$$|-k(\partial U/\partial r)| = |\rho L(dy/dt)|, \quad \text{at} \quad r = y(t). \qquad (3b)$$

### THE SOLUTION

We can obtain a closed form solution to $U(r, t)$ if we assume [2] that the acceleration $d^2y/dt^2$ is equal to $K(dy/dt)^3$, with $K$ = constant. Repeated integration of the assumed relation yields

$$y(t) = y_0 + (y_1^2/y_2)[1 - \sqrt{(1 - 2y_2t/y_1)}], \quad (4)$$

with $y_2 \equiv Ky_1^3$ = constant $\neq 0$. The constants $y_0$, $y_1$, $y_2$ are, respectively, the radius, velocity, and acceleration of the spherical solidification front at time $t = 0$. The initial radius $y_0$ is of course just equal to the radius of the spherical casting. Both $y_1$ and $y_2$ are arbitrary in magnitude, except that $y_2 \neq 0$.† Now define

$$B^2 = (-1/2\delta)y_1^3/y_2 \quad (5)$$

where $\delta$ is the thermal diffusivity. Then it may be verified by differentiation that the following [2] is the solution of equation (1) which satisfies the definition (2), the boundary condition (3b), and the equation-of-motion (4):

$$U(r, t) = U^* - (2L/c)\, B \exp[B^2]\, I(r, t), \qquad y_0 \geqslant r \geqslant y(t), \quad (6a)$$

where

$$I(r, t) = (r^*/r) \int_{Z = B(r - r^*)/(y - y^*)}^{Z = B} \exp[-Z^2]\, dZ + B^2(1 - r^*/r) \int_{Z = B(r - r^*)/(y - y^*)}^{Z = B} Z^{-2} \exp[-Z^2]\, dZ \quad (6b)$$

with (6c)

$$r^* \equiv y^* \equiv y_0 + y_1^2/y_2 = \text{constant}.$$

It may be noted that this solution was obtained without any boundary condition peing prescribed at the surface of the spherical casting, i.e. without any boundary condition being prescribed at $r = y_0 = y(0)$. We did not have the freedom to prescribe any condition at $r = y_0$ because we prescribed that the location of the solidification front be given by equation (4). The conditions at $r = y_0$ are fully determined by the solution (6).

## A CORRELATION OF DATA

The existence of the closed form solution (6) is a theoretical justification for any attempt to use the relation for $y(t)$ to correlate experimental data. Further, the solution (6) is of physical interest to the extent that the relation for $y(t)$ can be used to correlate experimental data. For correlation purposes it is convenient to put that relation in the form

$$[y(t)/y_0] = 1 + (1/p)\{1 - \sqrt{[1 - 2pq(t/y_0^2)]}\}, \quad (7a)$$

where

$$p \equiv y_0y_2/y_1^2 \quad \text{and} \quad q \equiv y_0y_1, \quad (7b)$$

so that

$$p/q = y_2/y_1^3 = K \quad \text{and} \quad 1/B^2 = -2\delta(p/q) = -2\delta K. \quad (7c)$$

The constants $p$ and $q$ can be evaluated by choosing two values of $y(t)/y_0$ for which $t/y_0^2$ is known. We choose two values which are representative of the data plotted by Schwartz [4] for the freezing of steel spheres. The two values are:

$$y/y_0 = 0{\cdot}80 \quad \text{at} \quad t/y_0^2 = 0{\cdot}15\ \text{min/in}^2, \quad (8a)$$

and

$$y/y_0 = 0{\cdot}50 \quad \text{at} \quad t/y_0^2 = 0{\cdot}50\ \text{min/in}^2. \quad (8b)$$

These values can be used with equation (7a) to show that

$$p = (20/7), \quad q = -(12/7)\ \text{in}^2/\text{min}, \quad (8c)$$

and

$$[y(t)/y_0] = 1 + (7/20)\{1 - \sqrt{[1 + (480/49)(t/y_0^2)(\text{in}^2/\text{min})]}\}. \quad (8d)$$

Equation (8d) is a good correlation of the data plotted by Schwartz [4]. As a check, we compute the time $T$ needed for a sphere to freeze all the way to the center. Setting $y(T) = 0$, we use equations (7a) and (8c) to obtain

$$T/y_0^2 = -(p/2 + 1)/q = (17/12)\ \text{min/in}^2 = 1{\cdot}42\ \text{min/in}^2. \quad (9a)$$

This result may be compared with the value obtainable from Chvorinov's 1939 correlation of experimental data, namely [4]

$$T/y_0^2 = [12{\cdot}5/(3)^2]\ \text{min/in}^2 = 1{\cdot}39\ \text{min/in}^2. \quad (9b)$$

The agreement as to freezing time is excellent.

† If $y_2 = 0$, we obtain the constant velocity case $y(t) = y_0 + y_1t$. The closed form solution for that case is available elsewhere [3].

## REFERENCES


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